Домашна работа 1 по ДИС 1, за спец. КН 1 курс, 2 поток. Вариант 2 за студентите с четна последна цифра от факултетния номер.

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Задача 1. Изследвайте функцията за обратимост. Намерете максимален интервал на обратимост, както и нейната обратна

$$f(x) = \log_5\left(x^2 - 2x + 1\right)$$

Решение:

$f(x) = \log_5\left(x^2 - 2x + 1\right) = \log_5\left((x-1)^2\right)$ Д.С. $x \neq 1$ (аргумент на логаритъм)

Разглеждаме за инективност. Нека $f(x_1) = f(x_2)$ $x_1, x_2 \in \mathbb{R} \setminus \{-1\}$

$\log_5\left((x_1-1)^2\right) = \log_5\left((x_2-1)^2\right) \iff (x_1-1)^2 = (x_2-1)^2 \iff |x_1-1| = |x_2-1|$

I. Случай: $x_1 - 1 > 0$ и $x_2 - 1 > 0$, $x_1 > 1$ $x_2 > 1$, $x_1, x_2 \in (1, +\infty)$

$\implies |x_1 - 1| = x_1 - 1$ и $|x_2 - 1| = x_2 - 1$

Тоест $|x_1 - 1| = |x_2 - 1| \iff x_1 - 1 = x_2 - 1 \implies x_1 = x_2$ има инекция

II. Случай: $x_1 - 1 < 0$ и $x_2 - 1 < 0$, $x_1 < 1$ $x_2 < 1$, $x_1, x_2 \in (-\infty, 1)$

$\implies |x_1 - 1| = 1 - x_1$ и $|x_2 - 1| = 1 - x_2$

Тоест $|x_1 - 1| = |x_2 - 1| \iff 1 - x_1 = 1 - x_2 \implies x_1 = x_2$ има инекция

Нека $y \in \mathbb{R}$. Тогава търсим x, такова че: $f(x) = y$

$y = f(x)$ $y = \log_5\left((x-1)^2\right)$ $5^y = (x-1)^2$ $\sqrt{5^y} = |x-1|$

I. Случай: $x \in (1, +\infty) \implies \sqrt{5^y} = x - 1$ $x = 1 + \sqrt{5^y}$

$x > 1$ $1 + \sqrt{5^y} > 1$ $\sqrt{5^y} > 0$ изпълнено за всяко y

II. Случай: $x \in (-\infty, 1) \implies \sqrt{5^y} = 1 - x$ $x = 1 - \sqrt{5^y}$

$x < 1$ $1 - \sqrt{5^y} < 1$ $-\sqrt{5^y} < 0$ $\sqrt{5^y} > 0$ изпълнено за всяко y

$f^{-1} : \mathbb{R} \to (-\infty, 1) \cup (1, +\infty)$ $f^{-1}(y) = 1 \pm \sqrt{5^y}$

Задача 2. Докажете тъждеството

$$\arcsin\left(2x\sqrt{1-x^2}\right) = \pi - 2\arccos(x), \text{ където } x \in \left[\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$$

Решение:

$$\arcsin\left(2x\sqrt{1-x^2}\right)=\alpha \iff \sin(\alpha)=2x\sqrt{1-x^2} \qquad \arccos(x)=\beta \iff \cos(\beta)=x$$

За $x=\dfrac{-\sqrt{2}}{2}$, $\quad \sin(\alpha)=2\cdot\dfrac{-\sqrt{2}}{2}\cdot\sqrt{1-\left(\dfrac{-\sqrt{2}}{2}\right)^2}=-\sqrt{2}\cdot\sqrt{\dfrac{1}{2}}=-1$, $\cos(\beta)=-\dfrac{\sqrt{2}}{2}$

Аналогично за $x=\dfrac{\sqrt{2}}{2}$, $\quad \sin(\alpha)=1$, $\quad \cos(\beta)=\dfrac{\sqrt{2}}{2}$

$\implies \alpha\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$, $\beta\in\left[-\dfrac{\pi}{4},\dfrac{\pi}{4}\right] \implies \sin(\beta)=+\sqrt{1-\cos^2(\beta)}=\sqrt{1-x^2}$

$$\sin(\alpha)=2x\sqrt{1-x^2}=2\cos(\beta)\sin(\beta)=\sin(2\beta)=\sin(\pi-2\beta) \qquad 2\beta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$\implies \alpha=\pi-2\beta$

Задача 3. Пресметнете границата на редицата

$$\lim_{n\to\infty}\frac{\sqrt[n-5]{n-5}+2^{2n}+n^{12}}{10^n-n^3+(-1)^{3n+5}}$$

Решение:

$$\lim_{n\to\infty}\frac{\sqrt[n-5]{n-5}+2^{2n}+n^{12}}{10^n-n^3+(-1)^{3n+5}}=\lim_{n\to\infty}\frac{2^{2n}\cdot\left(\frac{\sqrt[n-5]{n-5}}{2^{2n}}+1+\frac{n^{12}}{2^{2n}}\right)}{10^n\cdot\left(1-\frac{n^3}{10^n}+\frac{(-1)^{3n+5}}{10^n}\right)}=a_n$$

$$\lim_{n\to\infty}\frac{\sqrt[n-5]{n-5}}{2^{2n}}=\frac{\lim\limits_{n\to\infty}\sqrt[n-5]{n-5}}{\lim\limits_{n\to\infty}4^n}=\frac{1}{\infty}=0\,,\quad \lim_{n\to\infty}\frac{n^{12}}{2^{2n}}=0\,,\quad \lim_{n\to\infty}\frac{n^3}{10^n}=0\,,$$

$$\lim_{n\to\infty}\frac{(-1)^{3n+5}}{10^n}=\frac{\lim\limits_{n\to\infty}(-1)^{3n+5}}{\lim\limits_{n\to\infty}10^n}=\frac{\lim\limits_{n\to\infty}(-1)^{3n+5}}{\infty}=0 \quad ((-1)^{3n+5}\text{ е разходяща})$$

$$\implies a_n=\lim_{n\to\infty}\frac{2^{2n}}{10^n}=\lim_{n\to\infty}\frac{4^n}{10^n}=\lim_{n\to\infty}\left(\frac{2}{5}\right)^n \qquad \frac{2}{5}<1 \implies \lim_{n\to\infty}\left(\frac{2}{5}\right)^n=0$$

Задача 4. Пресметнете границата на редицата.

$$\lim_{n\to\infty}\left(\frac{n^2-5n+14}{n^2+1}\right)^{\frac{n^2-n+2}{n+5}}$$

Решение:

$$\lim_{n\to\infty}\left(\frac{n^2-5n+14}{n^2+1}\right)^{\frac{n^2-n+2}{n+5}}=\lim_{n\to\infty}\left(\frac{n^2+1-5n+13}{n^2+1}\right)^{\frac{n^2-n+2}{n+5}}=$$

$$=\lim_{n\to\infty}\left(1+\frac{13-5n}{n^2+1}\right)^{\frac{n^2-n+2}{n+5}}=\lim_{n\to\infty}\left(1+\frac{1}{\frac{n^2+1}{13-5n}}\right)^{\frac{n^2+1}{13-5n}\cdot\frac{13-5n}{n^2+1}\cdot\frac{n^2-n+2}{n+5}}$$

От $\lim\limits_{n\to\infty}\left(1+\frac{1}{p_n}\right)^{p_n b_n} = e^{\lim_{n\to\infty} b_n}$, задачата се свежда до $e^{\lim_{n\to\infty} b_n} = ?$, където $b_n = \frac{13-5n}{n^2+1} \cdot \frac{n^2-n+2}{n+5}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{13-5n}{n^2+1} \cdot \frac{n^2-n+2}{n+5} = \lim_{n\to\infty} \frac{13n^2-13n+26-5n^3+5n^2-10n}{n^3+5n^2+n+5} =$$

$$= \lim_{n\to\infty} \frac{-5n^3+18n^2-23n+26}{n^3+5n^2+n+5} = \lim_{n\to\infty} \frac{\cancel{n^3}\cdot\left(-5+\frac{18}{n}-\frac{23}{n^2}+\frac{26}{n^3}\right)}{\cancel{n^3}\cdot\left(1+\frac{5}{n}+\frac{1}{n^2}+\frac{5}{n^3}\right)} =$$

$$= \frac{\lim\limits_{n\to\infty} -5 + \lim\limits_{n\to\infty}\frac{18}{n} - \lim\limits_{n\to\infty}\frac{23}{n^2} + \lim\limits_{n\to\infty}\frac{26}{n^3}}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty}\frac{5}{n} + \lim\limits_{n\to\infty}\frac{1}{n^2} + \lim\limits_{n\to\infty}\frac{5}{n^3}} = \frac{-5+0+0+0}{1+0+0+0} = -5$$

$$\Longrightarrow e^{\lim_{n\to\infty} b_n} = e^{-5} = \frac{1}{e^5}$$